Teichmüller dynamics & hyperbolic surfaces

Joint work w/ James Farre

"measured laminations" "flat surfaces"

$$
\begin{array}{ccccc}
\mathrm{EQ} \circlearrowright \mathcal{P}\mathcal{M}_g & = & T^*\mathcal{M}_g & = & Q\mathcal{M}_g \circlearrowleft \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \\
\downarrow & & \downarrow & & \downarrow \\
\left\{ \begin{array}{c} \text{hyperbolic surfs} \\ \text{of genus } g \end{array} \right\} & = & \mathcal{M}_g & = & \left\{ \begin{array}{c} \text{Riemann surfaces} \\ \text{of genus } g \end{array} \right\}
\end{array}
$$

EQ & $\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ are <u>not</u> identical, but are...

[Mirzakhani '08]: the same, a.e.

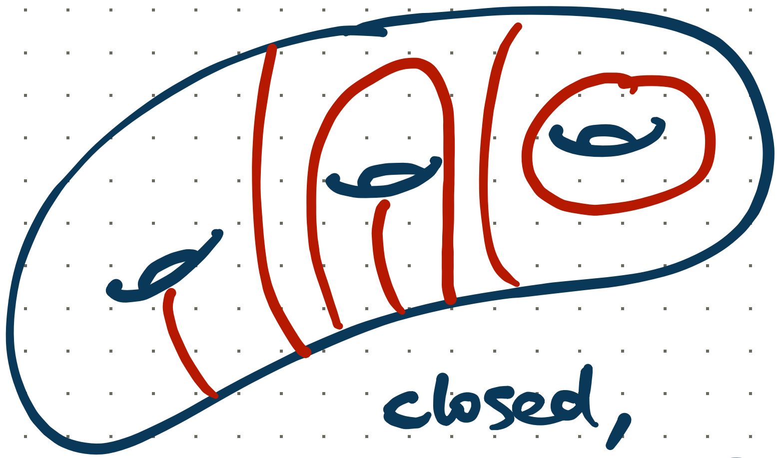
[C-Farre '21]: the same, measurably

[Arana-Herrera-Wright '22]: NOT the same, topologically

[C-Farre '24]: the same, w.r.t. equidistribution

[C-Farre '25]: the same, quantitatively

The earthquake flow

$S =$  closed, genus ≥ 2

P = pants decomposition

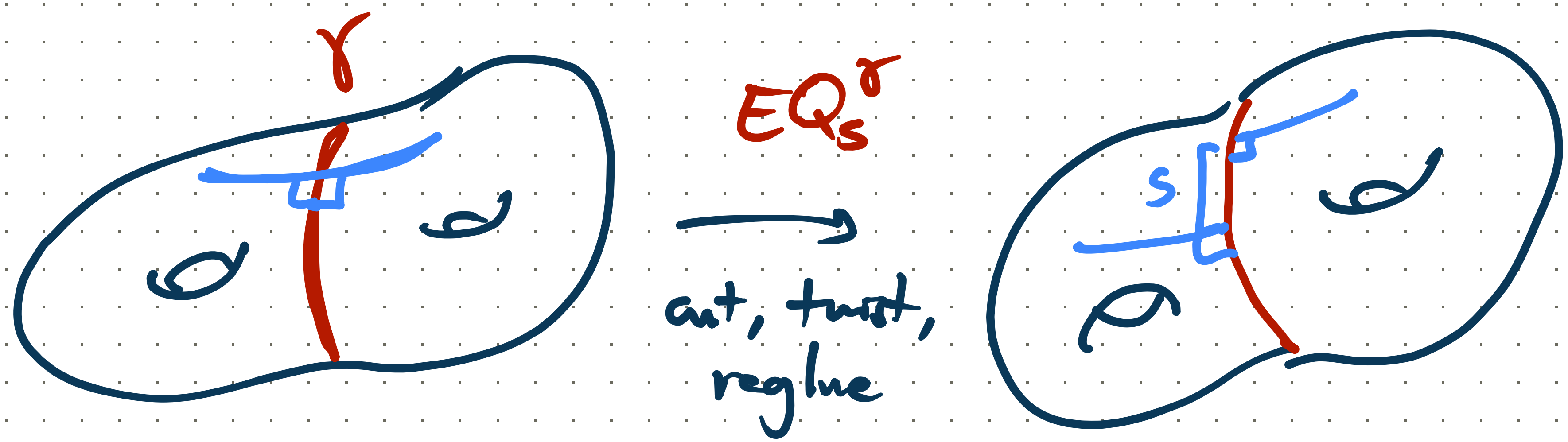
$$T_g \cong \underset{\text{(lengths)}\ \underline{l}}{\mathbb{R}^{3g-3}_{>0}} \times \underset{\text{(twists)}\ \underline{t}}{\mathbb{R}^{3g-3}}$$

$\omega = \sum dl \wedge dt$ WP symplectic form ⇝ μ_{WP} measure on $\mathcal{M}_g$

$$\begin{tikzcd}
PT_g \arrow[r] \arrow[d] & T_g = \tilde{\mathcal{M}}_g = \left\{ \pi_1 S \xrightarrow[\text{discrete}]{} PSL_2\mathbb{R} \right\}/\sim \arrow[d, "/\mathrm{Mod}_g"] \arrow[r, leftrightarrow] & G/K \arrow[d] & G \arrow[l] \arrow[d] \\
P\mathcal{M}_g \arrow[r] & \mathcal{M}_g = \{ \text{hyperbolic structures} \} \arrow[r, leftrightarrow] & \Gamma \backslash G/K & \Gamma \backslash G \arrow[l]
\end{tikzcd}$$

Simple closed curve $\gamma \rightsquigarrow$ length function $\ell_\gamma : T_g \to \mathbb{R}$.

$\rightsquigarrow$ Hamiltonian flow twisting γ



"measured laminations"

$$\mathrm{Mod}_g \backslash T_g \times \{\text{simple closed curves}\} \subset \mathrm{Mod}_g \backslash T_g \times \mathcal{ML} = \mathcal{P}\mathcal{M}_g$$

$$\mathcal{P}\mathcal{M}_g \;\; \pi \downarrow \;\; \mathcal{M}_g$$

<u>[Mirzakhani]</u> : $\exists$ ergodic measure ν on $\mathcal{P}^1\mathcal{M}_g$

s.t. $\pi_*(\nu) = B(X)\, \mu_{WP}$.

What is this good for?

Thm [Mirzakhani, '08]

For any closed hyperbolic X of genus g,

$$\#\left\{\begin{array}{c}\text{simple closed geodesics} \\ \text{on } X \text{ w/ length} \le L\end{array}\right\} \sim c_{scc}\, B(X)\, L^{6g-6}$$

topological constant

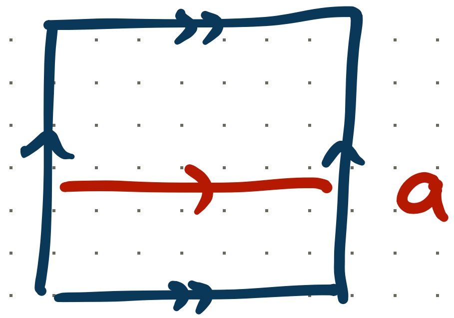
$$\#\{\text{non separating} \ldots\} \sim c_{nonsep}\, B(X)\, L^{6g-6}$$

$$\#\left\{\begin{array}{c}\text{pants decompositions w/} \\ \text{total length} \le L\end{array}\right\} \sim c_{pants}\, B(X)\, L^{6g-6}$$

[Arana-Herrera, Liu]: max length $\le L \sim c_{max}\, B(X)\, L^{6g-6}$

Equidistribution & Counting Problems

$$\#\{v \in \mathbb{Z}^2_{prim} \mid \|v\| \le L\} \sim \frac{6}{\pi} L^2$$

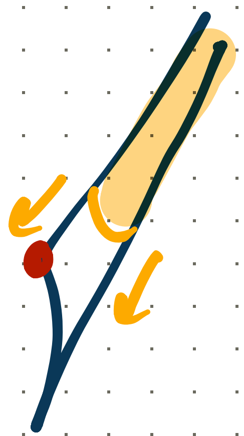


$$\mathbb{Z}^2_{prim} = SL_2\mathbb{Z} \cdot a$$

$$\|ga\|_{\mathbb{Z}^2} \le L \longleftrightarrow \|a\|_{g^{-1}\mathbb{Z}^2} \le L$$

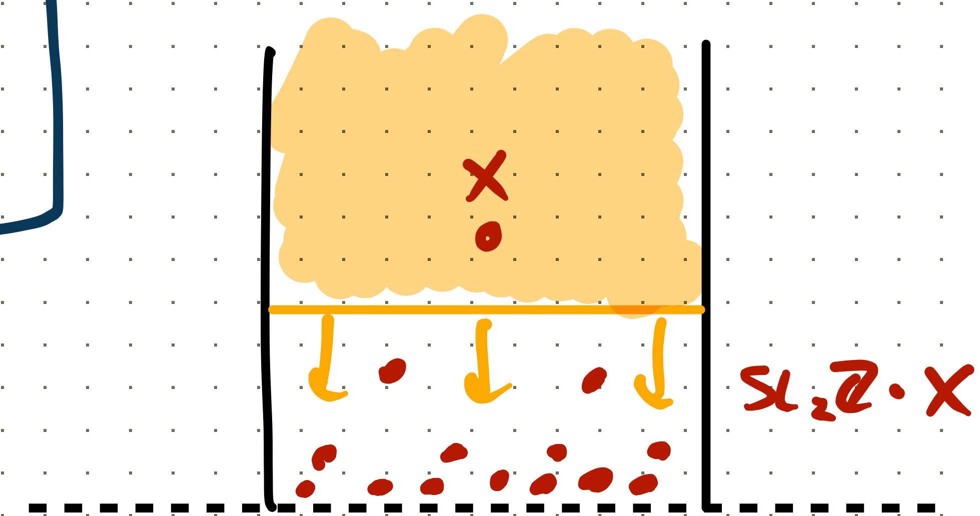
$$\#\{\|v\|_{\mathbb{Z}^2} \le L\} = \#\{g^{-1}\mathbb{Z}^2 \in \text{length} \le L \text{ horoball}\}$$

in $\mathbb{H}^2/\mathrm{Stab}(a)$



Expanding horoballs equidistribute in $\mathbb{H}^2/SL_2\mathbb{Z}$

$$\#\{g^{-1}X\} \sim \frac{\mathrm{vol}(\text{horoball})}{\mathrm{vol}(\mathbb{H}^2/SL_2\mathbb{Z})}$$



Equidistribution & Counting Problems

$$\#\{\text{scc } \gamma \text{ on } X \text{ w/ length } \leq L\} \sim c_{scc} B(X) L^{6g-6}$$

$v \in \mathbb{Z}^2_{prim} \rightsquigarrow$ scc γ

$SL_2\mathbb{Z} \rightsquigarrow \mathrm{Mod}(S)$

$\mathbb{H}^2 / SL_2\mathbb{Z} \rightsquigarrow \mathcal{M}_g$

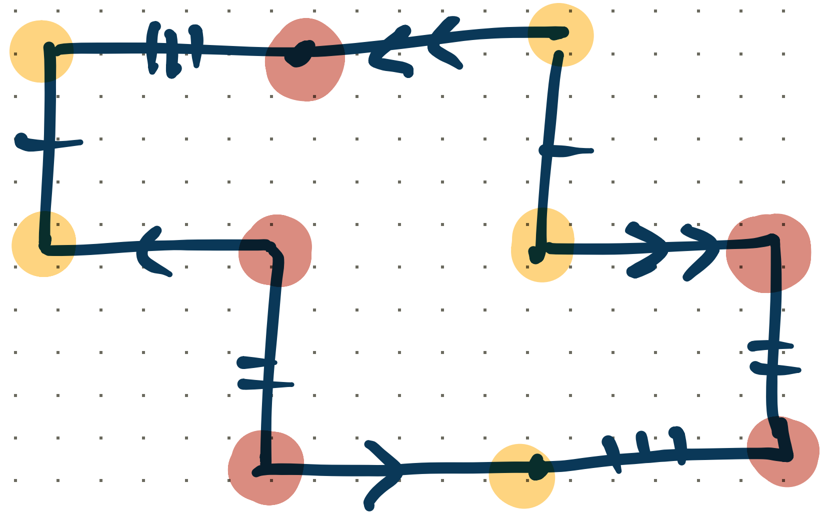
hyperbolic vol $\rightsquigarrow B(X) \cdot \mu_{wp}$

horoballs in $\mathbb{H}^2 \rightsquigarrow \underbrace{\{X \in T_g : \ell_X(\gamma) \leq L\}}_{H_L}$

Key result is equidistribution of expanding horoballs!
(in $\mathcal{P}^1\mathcal{M}_g$)

The horocycle flow

Quadratic differentials



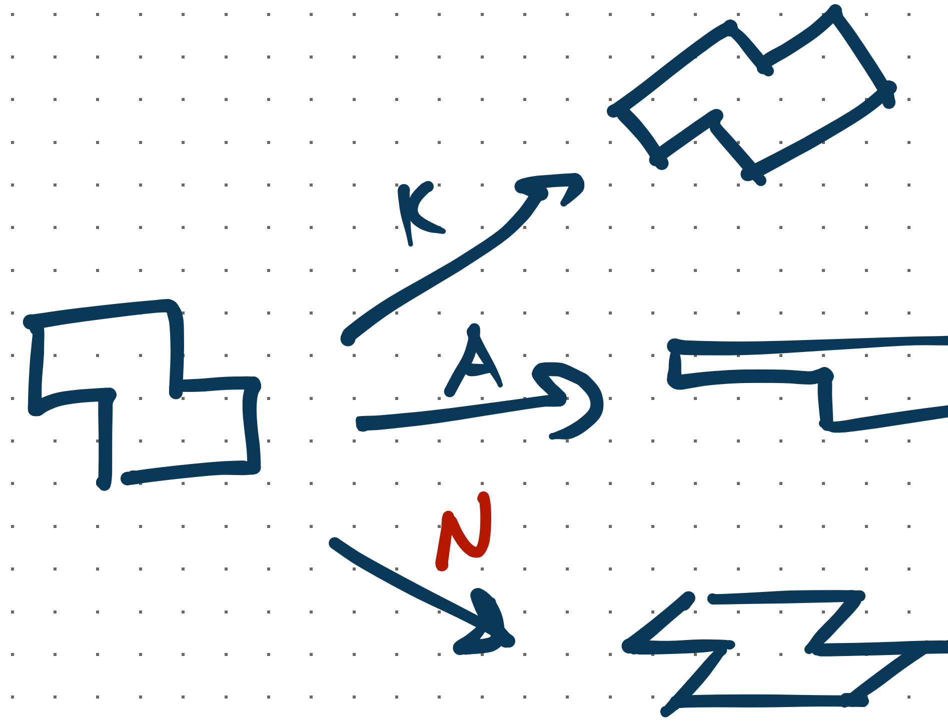
translations are holomorphic
$\Rightarrow$ get a Riemann surface
+ flat metric w/ cone pts

$\{$unit area quadratic differentials$\} =: Q^1\mathcal{M}_g$

- Stratified by # & order of cone pts
- stratum $\underset{\text{locally}}{\overset{\text{"period coords"}}{\sim}} \mathbb{C}^N$

"MSV measure" $\leftsquigarrow$ Lebesgue

$SL_2\mathbb{R} \curvearrowright$ stratum



N = "horocycle flow"

$P = AN = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$

$$\mathbb{C}^N = \underbrace{\mathbb{R}^N}_{\text{expanded by } g_t} \oplus \underbrace{i\,\mathbb{R}^N}_{\text{contracted}} \rightsquigarrow \text{(un)stable foliations}$$

(Forni: NUH wrt MV)

Thm: [Masur, Veech]

$SL_2\mathbb{R}$ is ergodic on every stratum. ($\Rightarrow N$ is)

Magic Wand Theorem: [Eskin–Mirzakhani; Eskin–Mirzakhani–Mohammadi; Filip]

$\forall q \in \mathcal{QM}_g$,

① $\overline{P_q} = \overline{SL_2\mathbb{R}\, q}$ is a special subvariety of $\mathcal{QM}_g$

② Every P-invariant measure is $SL_2\mathbb{R}$-invariant,
& is Lebesgue on such a subvariety.

③ P_q equidistributes to Lebesgue on $\overline{P_q}$

From hyperbolic to flat

(& back again)

EQ earthquake flow $\circlearrowright \mathcal{P}^1\mathcal{M}_g \longleftrightarrow Q^1\mathcal{M}_g \circlearrowleft$ N horocycle flow

<u>[Mirzakhani '08]</u>: $\exists$ mble map defined a.e. (w.r.t. Leb) conjugating EQ & N.

⚠ <u>not</u> continuous!

$\Rightarrow$ EQ is ergodic w.r.t. Lebesgue

<u>[C-Farre '21]</u>: $\exists$ mble <u>bijection</u> (Borel isomorphism)

$\Rightarrow$ many new ergodic measures

<u>[Arana-Herrera–Wright '22]</u>: $\nexists$ topological conjugacy.

$\Rightarrow$ can't just pull back magic wand...

EQ earthquake flow $\circlearrowright \mathcal{P}^1\mathcal{M}_g \longleftrightarrow Q^1\mathcal{M}_g \circlearrowleft$ N horocycle flow

[C-Farre '24]: You CAN pull back equidistribution to nice measures.

ie. if $\nu_n \to \nu$ on $Q^1\mathcal{M}_g$, $\mathcal{O}^*\nu_n \to \mathcal{O}^*\nu$ on $\mathcal{P}^1\mathcal{M}_g$.
$SL_2\mathbb{R}$-inv^t

Eg: Expanding unstables equid. in $Q^1\mathcal{M}_g$ [EMM, Forni]
$\Rightarrow$ expanding horoballs equid in $\mathcal{P}^1\mathcal{M}_g$!

[C-Farre, soon]: You can even pull back effective equidistribution/mixing results.
$\Rightarrow \mathcal{O}^{-1} g_t \mathcal{O}$ is exponentially mixing on $\mathcal{P}^1\mathcal{M}_g$

Other applications:

- P orbits ("stretchquake disks") equid. in $\mathbb{P}^1\mathcal{M}_g$.

  (magic wand)

- $\exists$ nongeneric orbits of EQ flow

  (Chaika-Khalil-Smillie)

- Mirzakhani's "twist torus conjecture"

  (classification of high rank subvarieties)

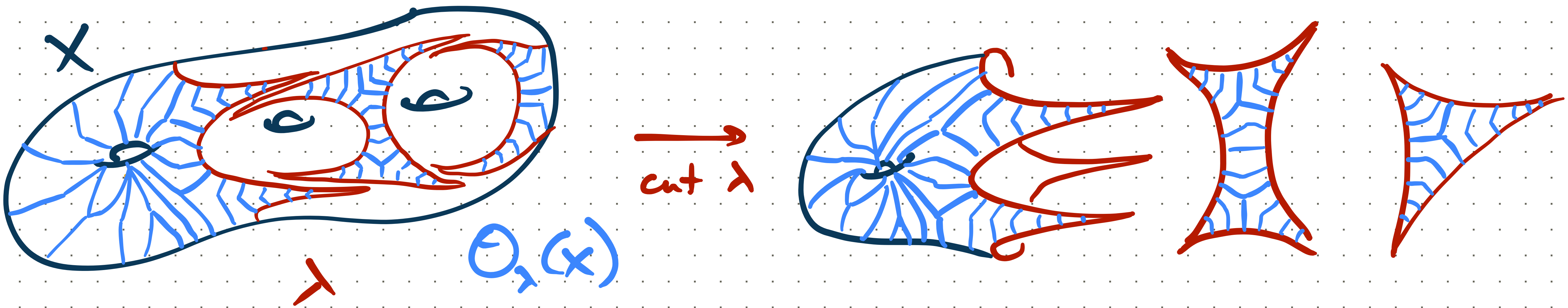
- New pf of effective scc counting

  (first proved by Eskin-Mirzakhani-Mohammadi)

[Gardiner–Masur]

$$\mathcal{P}T_g = T_g \times \mathcal{ML} \xrightarrow{\Theta} \underset{\mathbb{R}}{\mathcal{ML}} \times \underset{i\mathbb{R}}{\mathcal{ML}} \setminus \Delta = \mathcal{Q}T_g$$

Idea: Build map unstable by unstable $T_g \to \mathcal{ML}$



[Thurston, Bonahon]: λ max $\Rightarrow$ $\Theta_\lambda : T_g \cong \mathcal{ML} \setminus \Delta_\lambda$

[CF 21]: True for general λ

[CF 24]: $\Theta^{\pm 1}$ cts as λ varies in Hausdorff topology

[CF 24]: $\Theta^{\pm 1}$ cts as λ varies in Hausdorff topology

$\Rightarrow$ Θ^{-1} cts a.e. wrt. nice measures ν

$\Rightarrow (\Theta^{-1})_*$ preserves convergence to ν!

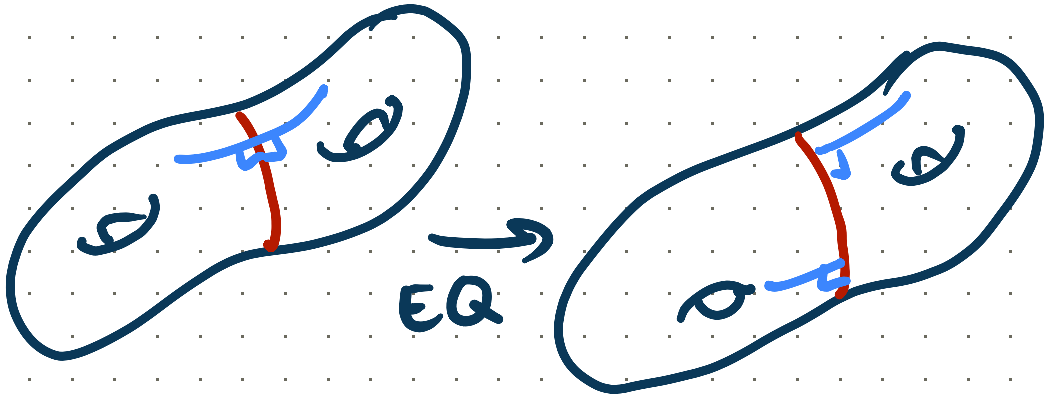
[CF 25]: For $f \in C_c(\mathbb{P}^1\mathcal{M}_g)$ & Lipschitz,

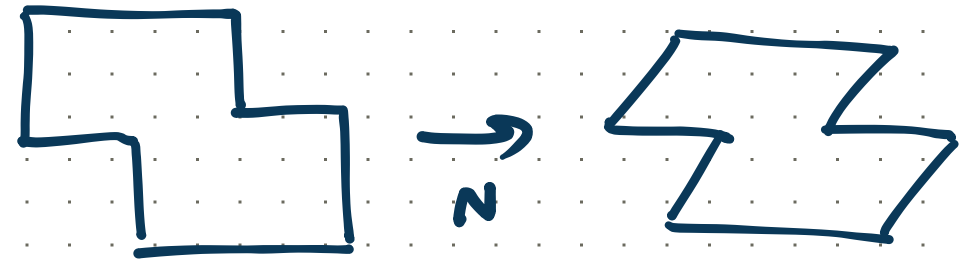
build nice smoothings F of $(\Theta^{-1})^* f$ w/

1. $\| F - \Theta^{-1*} f \|_1$ small
2. $\| F \|_{Lip} / \| f \|_{Lip}$ slowly growing

$\Rightarrow$ can pull back exponential mixing of g_t!

(Avila-Gouëzel-Yoccoz, Avila-Resende, Avila-Gouëzel)





$$\mathrm{EQ} \circlearrowright \mathcal{PM}_g \xrightarrow{\Theta} \mathcal{QM}_g \circlearrowleft N$$

[M]: measurably isomorphic wrt Leb

[CF]: Borel isomorphic

[AHW]: $\nexists$ topological conjugacy

[CF]: Pull back equidistribution

[CF]: Pull back quantitative results

<u>Lemma</u>: $\mu_n \xrightarrow{w^*} \mu$ on X. $f: X \to Y$ continuous μ-a.e.

Then $f_* \mu_n \xrightarrow{w^*} f_* \mu$.

Portmanteau

<u>Pf</u>: $\forall$ closed A,

$$\begin{array}{ccc} \limsup f_* \mu_n(A) & \overset{\text{WTS}}{\leq} & f_* \mu(A) \\ \| & & \| \\ \limsup \mu_n(f^{-1}(A)) & & \mu(f^{-1}(A)) \\ \text{IV} & & \| \\ \limsup \mu_n(\overline{f^{-1}(A)}) & \underset{\mu_n \xrightarrow{*} \mu}{\leq} & \mu(\overline{f^{-1}(A)}) \end{array}$$

$\square$

$x \in \overline{f^{-1}(A)} \setminus f^{-1}(A) \Rightarrow \exists x_n \to x$ but $f(x_n) \not\to f(x)$

$\Rightarrow$ discontinuous @ x.